Homework 10 Solutions

①

(a) T is linear.

Let $(a,b),(x,y)$ be in $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$T((a,b)+(x,y)) = T(a+x, b+y) = ((a+x)+2(b+y), 3(a+x)-(b+y))$$
$$= ((a+2b)+(x+2y), (3a-b)+(3x-y))$$
$$= (a+2b, 3a-b) + (x+2y, 3x-y) = T(a,b)+T(x,y)$$

and $T(\alpha(a,b)) = T(\alpha a, \alpha b) = (\alpha a + 2\alpha b, 3\alpha a - \alpha b)$
$$= \alpha(a+2b, 3a-b) = \alpha T(a,b).$$

(b) T is linear.

Let (a,b,c) and (x,y,z) be in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$T((a,b,c)+(x,y,z)) = T(a+x, b+y, c+z)$$
$$= (2(a+x)-(b+y)+(c+z), (b+y)-4(c+z))$$
$$= ((2a-b+c)+(2x-y+z), (b-4c)+(y-4z))$$
$$= (2a-b+c, b-4c) + (2x-y+z, y-4z)$$
$$= T(a,b,c)+T(x,y,z)$$

and

$$T(\alpha(a,b,c)) = T(\alpha a, \alpha b, \alpha c) = (2\alpha a - \alpha b + \alpha c, \alpha b - 4\alpha c)$$
$$= \alpha(2a-b+c, b-4c) = \alpha T(a,b,c)$$

(c) T is not linear. For example, let $\vec{v}_1 = (1,1)$ and $\vec{v}_2 = (-1,-1)$. Then

$$T(\vec{v}_1+\vec{v}_2) = T((1,1)+(-1,-1)) = T(0,0) = \sqrt{0^2+0^2}$$
$$= 0$$

but

$$T(\vec{v}_1)+T(\vec{v}_2) = T(1,1)+T(-1,-1) =$$
$$= \sqrt{1^2+1^2} + \sqrt{(-1)^2+(-1)^2}$$
$$= \sqrt{2}+\sqrt{2} = 2\sqrt{2},$$

So, $T(\vec{v}_1+\vec{v}_2) \neq T(\vec{v}_1)+T(\vec{v}_2)$.

T is linear:

(d) Let $M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M_2 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be in $M_{2,2}$

and $\alpha \in \mathbb{R}$.

Then

$$T(M_1+M_2) = T\begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix} = 3(a+x)-4(b+y) + (c+z)-(d+w)$$

$$= (3a-4b+c-d)+(3x-4y+z-w)$$
$$= T\begin{pmatrix} a & b \\ c & d \end{pmatrix} + T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = T(M_1)+T(M_2)$$

and

$$T(\alpha M_1) = T\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} = 3(\alpha a)-4(\alpha b)+(\alpha c)-(\alpha d)$$
$$= \alpha[3a-4b+c-d]$$
$$= \alpha T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha T(M_1)$$

(e) T is not linear. For example,

$$T\begin{pmatrix}1&1\\5&3\end{pmatrix}+T\begin{pmatrix}1&0\\0&1\end{pmatrix}=(1^2+1^2)+(1^2+0^2)=3$$

but

$$T\left(\begin{pmatrix}1&1\\5&3\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)=T\begin{pmatrix}2&1\\5&4\end{pmatrix}=2^2+1^2=5$$

and $5\neq 3$.

(f) T is linear.

Let $f_1=a+bx+cx^2$ and $f_2=d+ex+gx^2$ be in P_2 and $\alpha\in\mathbb{R}$. Then

$$\begin{aligned}T(f_1+f_2)&=T((a+d)+(b+e)x+(c+g)x^2)\\&=(a+d)+(b+e)(1+x)+(c+g)(1+x)^2\\&=[a+b(1+x)+c(1+x^2)]+[d+e(1+x)+g(1+x)^2]\\&=T(f_1)+T(f_2)\end{aligned}$$

and

$$\begin{aligned}T(\alpha f_1)&=T(\alpha a+\alpha bx+\alpha cx^2)=(\alpha a)+(\alpha b)(1+x)+(\alpha c)(1+x)^2\\&=\alpha[a+b(1+x)+c(1+x)^2]\\&=\alpha T(f_1)\end{aligned}$$

(g) T is not linear.

For example, let $f = 1 + x + x^2$ and $\alpha = 2$.

Then

$$\alpha T(f) = 2T(1 + x + x^2) = 2[(1+1) + (1+1)x + (1+1)x^2]$$
$$= 2[2 + 2x + 2x^2] = 4 + 4x + 4x^2$$

but

$$T(\alpha f) = T(2 + 2x + 2x^2)$$
$$= (1+2) + (1+2)x + (1+2)x^2$$
$$= 3 + 3x + 3x^2$$

So, $\alpha T(f) \neq T(\alpha f)$ in this example.

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(2) Let (x, y, z) be in $\mathbb{R}^3$ and T be linear with $T(1,0,0) = (2,1,-1)$, $T(0,1,0) = (0, \pi, 2/3)$, and $T(0,0,1) = (-1,0,0)$

Then, since T is linear we have that

$$T(x,y,z) = T(x,0,0) + T(0,y,0) + T(0,0,z)$$
$$= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(2,1,-1) + y(0,\pi,2/3) + z(-1,0,0)$$
$$= (2x - z, x + \pi y, -x + \tfrac{2}{3} y).$$

③

(a) $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix}+0\begin{pmatrix}1\\2\end{pmatrix}$

$T\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix}=0\begin{pmatrix}1\\1\end{pmatrix}+3\begin{pmatrix}1\\2\end{pmatrix}$

So, $[T]_\beta=\left(\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_\beta \middle| \left[T\begin{pmatrix}1\\2\end{pmatrix}\right]_\beta\right)=\begin{pmatrix}2&0\\0&3\end{pmatrix}$.

(b) We start with $\vec{x}=\begin{pmatrix}1\\0\end{pmatrix}$.

Then $\vec{x}=\begin{pmatrix}1\\0\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix}-1\cdot\begin{pmatrix}1\\2\end{pmatrix}$

So, $[\vec{x}]_\beta=\begin{pmatrix}2\\-1\end{pmatrix}$

And $T(\vec{x})=T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}=4\begin{pmatrix}1\\1\end{pmatrix}-3\begin{pmatrix}1\\2\end{pmatrix}$

$$\boxed{\begin{aligned}&\begin{pmatrix}1\\-2\end{pmatrix}=a\begin{pmatrix}1\\1\end{pmatrix}+b\begin{pmatrix}1\\2\end{pmatrix}\\&\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}a+b\\a+2b\end{pmatrix}\Rightarrow \begin{matrix}1=a+b\\-2=a+2b\end{matrix}\\&\Downarrow\\&3=-b\\&b=-3\\&a=4\end{aligned}}$$

So, $[T(\vec{x})]_\beta=\begin{pmatrix}4\\-3\end{pmatrix}$.

And,

$[T]_\beta[x]_\beta=\begin{pmatrix}2&0\\0&3\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}4\\-3\end{pmatrix}=[T(\vec{x})]_\beta$.

Now let $\vec{x}=\begin{pmatrix}2\\1\end{pmatrix}$.

Then $\vec{x}=\begin{pmatrix}2\\1\end{pmatrix}=3\begin{pmatrix}1\\1\end{pmatrix}-1\begin{pmatrix}1\\2\end{pmatrix}$. So, $[x]_\beta=\begin{pmatrix}3\\-1\end{pmatrix}$

$$\begin{aligned}&\begin{pmatrix}2\\1\end{pmatrix}=a\begin{pmatrix}1\\1\end{pmatrix}+b\begin{pmatrix}1\\2\end{pmatrix}\\&\left.\begin{matrix}2=a+b\\1=a+2b\end{matrix}\right\}\begin{matrix}a=3\\b=-1\end{matrix}\end{aligned}$$

And $T(x)=T\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix}=6\begin{pmatrix}1\\1\end{pmatrix}-3\begin{pmatrix}1\\2\end{pmatrix}$

So, $[T(x)]_\beta=\begin{pmatrix}6\\-3\end{pmatrix}$. And,

$[T]_\beta[x]_\beta=\begin{pmatrix}2&0\\0&3\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}6\\-3\end{pmatrix}=[T(x)]_\beta$

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(a)

$$\left.\begin{array}{l} T(1) = 1 = 1\cdot 1 + 0\cdot x \\ T(x) = 1-2x = 1\cdot 1 - 2\cdot x \\ T(x^2) = 3x = 0\cdot 1 + 3\cdot x \end{array}\right\} \text{So, } [T]_\beta^{\beta'} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix}$$

(b)

First we look at $\vec{v} = 1+x+x^2$.

Here, $[\vec{v}]_\beta = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. And, $T(\vec{v}) = T(1+x+x^2) = 2+x$

So, $[T(\vec{v})]_{\beta'} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

And, $[T]_\beta^{\beta'}[\vec{v}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = [T(\vec{v})]_{\beta'}$

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Now let $\vec{v} = 2x^2$.

Then, $[\vec{v}]_\beta = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$. And $T(\vec{v}) = T(2x^2)$

$$= 0 - (0 - 3\cdot 2)x$$
$$= 6x = 0 + 6x$$

So, $[T(\vec{v})]_{\beta'} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

And, $[T]_\beta^{\beta'}[\vec{v}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} = [T(\vec{v})]_{\beta'}$.

(5)

(a) $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}-1\\0\end{pmatrix}$

$T\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix} = -1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}-1\\0\end{pmatrix}$.

So, $[T]_\beta = \begin{pmatrix}2 & -1\\2 & 0\end{pmatrix}$

(b) We first let $\vec{v} = \begin{pmatrix}1\\0\end{pmatrix}$.

Then $\vec{v} = \begin{pmatrix}1\\0\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} - 1\cdot\begin{pmatrix}-1\\0\end{pmatrix}$. So, $[v]_\beta = \begin{pmatrix}0\\-1\end{pmatrix}$

And $T(\vec{v}) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}-1\\0\end{pmatrix}$.

So, $[T(\vec{v})]_\beta = \begin{pmatrix}1\\0\end{pmatrix}$.

And, $[T]_\beta[\vec{v}]_\beta = \begin{pmatrix}2 & -1\\2 & 0\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = [T(\vec{v})]_\beta$.

Now let $\vec{v} = \begin{pmatrix}0\\1\end{pmatrix}$.

Then $\vec{v} = \begin{pmatrix}0\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 1\cdot\begin{pmatrix}-1\\0\end{pmatrix}$. So, $[v]_\beta = \begin{pmatrix}1\\1\end{pmatrix}$.

And, $T(\vec{v}) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}-1\\0\end{pmatrix}$

So, $[T(\vec{v})]_\beta = \begin{pmatrix}1\\2\end{pmatrix}$.

And, $[T]_\beta[\vec{v}]_\beta = \begin{pmatrix}2 & -1\\2 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2-1\\2+0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$= [T(\vec{v})]_\beta$

$$\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}-3\\4\end{pmatrix} \leftrightarrow \begin{matrix}1 = 2a-3b\\0 = a+4b\end{matrix} \leftrightarrow \begin{matrix}b = -1/11\\a = 4/11\end{matrix}$$

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(a)
$$\left.\begin{matrix} T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{4}{11}\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{11}\begin{pmatrix}-3\\4\end{pmatrix} \\ T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix} = -1\cdot\begin{pmatrix}2\\1\end{pmatrix} + 0\begin{pmatrix}-3\\4\end{pmatrix}\end{matrix}\right\} \text{So, } [T]_\beta^{\beta'} = \begin{pmatrix}\frac{4}{11} & -1\\ -\frac{1}{11} & 0\end{pmatrix}$$

(b)
$$\left.\begin{matrix} I\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \frac{4}{11}\begin{pmatrix}2\\1\end{pmatrix} - \frac{1}{11}\begin{pmatrix}-3\\4\end{pmatrix} \\ I\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = \frac{3}{11}\begin{pmatrix}2\\1\end{pmatrix} + \frac{2}{11}\begin{pmatrix}-3\\4\end{pmatrix}\end{matrix}\right\} \text{So, } Q = [I]_\beta^{\beta'} = \begin{pmatrix}4/11 & 3/11\\ -1/11 & 2/11\end{pmatrix}.$$

$$\begin{pmatrix}0\\1\end{pmatrix} = a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}-3\\4\end{pmatrix}$$
$$\left.\begin{matrix}0 = 2a-3b\\1 = a+4b\end{matrix}\right\} \begin{matrix}b = 2/11\\a = 3/11\end{matrix}$$

$$\begin{pmatrix}1\\1\end{pmatrix} = a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}-3\\4\end{pmatrix} \leftrightarrow \begin{matrix}1 = 2a-3b\\1 = a+4b\end{matrix} \leftrightarrow \begin{matrix}b = 1/11\\a = 7/11\end{matrix}$$

(c) Let $\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$.

Then $\vec{v} = \begin{pmatrix}1\\1\end{pmatrix} = \frac{7}{11}\begin{pmatrix}2\\1\end{pmatrix} + \frac{1}{11}\begin{pmatrix}-3\\4\end{pmatrix}$. So, $[\vec{v}]_{\beta'} = \begin{pmatrix}7/11\\1/11\end{pmatrix}$

And $[\vec{v}]_\beta = \begin{pmatrix}1\\1\end{pmatrix}$.

And, $Q[\vec{v}]_\beta = Q\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4/11 & 3/11\\ -1/11 & 2/11\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}7/11\\1/11\end{pmatrix} = [\vec{v}]_{\beta'}$

Now let $\vec{v} = \begin{pmatrix}-1\\5\end{pmatrix}$.

Then $\vec{v} = \begin{pmatrix}-1\\5\end{pmatrix} = 1\cdot\begin{pmatrix}2\\1\end{pmatrix} + 1\cdot\begin{pmatrix}-3\\4\end{pmatrix}$. So, $[\vec{v}]_{\beta'} = \begin{pmatrix}1\\1\end{pmatrix}$.

$$\begin{pmatrix}-1\\5\end{pmatrix} = a\begin{pmatrix}2\\1\end{pmatrix} + b\begin{pmatrix}-3\\4\end{pmatrix}$$
$$\left.\begin{matrix}-1 = 2a-3b\\5 = a+4b\end{matrix}\right\} \begin{matrix}a = 1\\b = 1\end{matrix}$$

And $[\vec{v}]_\beta = \begin{pmatrix}-1\\5\end{pmatrix}$.

And, $Q[\vec{v}]_\beta = \begin{pmatrix}4/11 & 3/11\\ -1/11 & 2/11\end{pmatrix}\begin{pmatrix}-1\\5\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

$= [\vec{v}]_{\beta'}$

(d) From b we have that $Q = \begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix}$.

So, $$Q^{-1} = \frac{1}{\frac{4}{11}\cdot\frac{2}{11} - \frac{3}{11}\left(-\frac{1}{11}\right)} \begin{pmatrix} 2/11 & -3/11 \\ 1/11 & 4/11 \end{pmatrix} = \frac{1}{\left(\frac{11}{11^2}\right)} \begin{pmatrix} 2/11 & -3/11 \\ 1/11 & 4/11 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

And

$$\left.\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{So, } [T]_\beta = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix},$$

And

$$\boxed{\begin{pmatrix} 0 \\ -1 \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} -3 \\ 4 \end{pmatrix} \Leftrightarrow \begin{matrix} 0 = 2a - 3b \\ -1 = a + 4b \end{matrix} \Leftrightarrow \begin{matrix} b = -2/11 \\ a = -3/11 \end{matrix}}$$

$$\left.\begin{aligned} T\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{3}{11}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{2}{11}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ T\begin{pmatrix} -3 \\ 4 \end{pmatrix} &= \begin{pmatrix} -11 \\ -4 \end{pmatrix} = -\frac{56}{11}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{3}{11}\begin{pmatrix} -3 \\ 4 \end{pmatrix} \end{aligned}\right\} \text{So, } [T]_{\beta'} = \begin{pmatrix} -\frac{3}{11} & -\frac{56}{11} \\ -\frac{2}{11} & \frac{3}{11} \end{pmatrix},$$

$$\boxed{\begin{aligned} &\begin{pmatrix} -11 \\ -4 \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} -3 \\ 4 \end{pmatrix} \\ &\left.\begin{matrix} -11 = 2a - 3b \\ -4 = a + 4b \end{matrix}\right\} \begin{matrix} b = 3/11 \\ a = -56/11 \end{matrix} \end{aligned}}$$

Here we go!

$$Q^{-1}[T]_{\beta'}Q = Q^{-1}\begin{pmatrix} -3/11 & -56/11 \\ -2/11 & 3/11 \end{pmatrix}\begin{pmatrix} 4/11 & 3/11 \\ -1/11 & 2/11 \end{pmatrix} = Q^{-1}\begin{pmatrix} 4/11 & -1 \\ -1/11 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 4/11 & -1 \\ -1/11 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} = [T]_\beta$$

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(a)
$$\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\\0\end{pmatrix} = -1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + 0\begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\begin{pmatrix}1\\0\\0\end{pmatrix} - 1\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

So, $[I]_{\beta}^{\beta'} = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}$

$$\boxed{\left.\begin{array}{l}2=a-b\\ 1=a+3b\end{array}\right\}\begin{array}{l}b=-1/4\\ a=7/4\end{array}}$$

(b)
$$\begin{pmatrix}2\\1\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}-1\\3\end{pmatrix} = \tfrac{7}{4}\begin{pmatrix}1\\1\end{pmatrix} - \tfrac{1}{4}\begin{pmatrix}-1\\3\end{pmatrix}$$
$$\begin{pmatrix}-1\\1\end{pmatrix} = c\begin{pmatrix}1\\1\end{pmatrix} + d\begin{pmatrix}-1\\3\end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}-1\\3\end{pmatrix}$$

$$\boxed{\begin{array}{l}d=1/2\\ c=-1/2\end{array}\left\{\begin{array}{l}-1=c-d\\ 1=c+3d\end{array}\right.}$$

So, $[I]_{\beta}^{\beta'} = \begin{pmatrix}7/4 & -1/2\\ -1/4 & 1/2\end{pmatrix}$

8 (a)

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \text{ So, } [\vec{v}]_\beta = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

And, $$\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \text{ So, } [\vec{v}]_{\beta'} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

And, $$Q[\vec{v}]_\beta = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = [\vec{v}]_{\beta'}$$

$$\boxed{\left.\begin{matrix} -1 = 2a - b \\ 2 = a + b \end{matrix}\right\} \begin{matrix} b = 5/3 \\ a = 1/3 \end{matrix}}$$

(b) $$\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}. \text{ So, } [\vec{v}]_\beta = \begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix}.$$

And
$$\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1/4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3/4\begin{pmatrix} -1 \\ 3 \end{pmatrix}. \text{ So, } [\vec{v}]_{\beta'} = \begin{pmatrix} -1/4 \\ 3/4 \end{pmatrix}.$$

$$\boxed{\left.\begin{matrix} -1 = a + (-b) \\ 2 = a + 3b \end{matrix}\right\} \begin{matrix} b = 3/4 \\ a = -1/4 \end{matrix}}$$

And,
$$Q[\vec{v}]_\beta = \begin{pmatrix} 7/4 & -1/2 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 1/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 3/4 \end{pmatrix} = [\vec{v}]_{\beta'}.$$